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The Hopkins-Neeman theorem

Original thm: perfect complexes over an (ungraded) comm.-noeth. ring.
Here:

The Statement

S a graded commutative noeth.-alg ($S = k[x_1, \dots, x_r]$, $|x_i| = 1, 2$).

two conventions: $\to$ $a \cdot b = b \cdot a$ comm. graded

$\to$ $a \cdot b = (-1)^{|a| \cdot |b|} b \cdot a$ graded-commutative

- $\rightsquigarrow$ Assume either $S = S^{even}$ or $\operatorname{char} k = 2$ $\rightsquigarrow$ 1st situation.
- View S as a DGA with $d = 0$. Concretely:

$$\cdots \xrightarrow{0} S^{-1} \xrightarrow{0} S^0 \xrightarrow{0} S^1 \xrightarrow{0} \cdots$$

- DG-modules, $M \in \mathrm{Mod}_{DG} S$ (here: right modules)
  - graded S-module
  - $d : M^i \to M^{i+1}$
  - Leibniz rule: $d(m \cdot s) = d(m) \cdot s + \underbrace{(-1)^{|m|} m\, \cancel{d(s)}}_{\text{disappears!}}$ ($d(s) = 0$)
- Special case: If $S = S^0$, then $\mathrm{Mod}_{DG}(S) \overset{(\sim)}{=} \mathcal{C}(S)$, chain complexes!
- Last talk: $\mathcal{D}(S) = \mathrm{Mod}_{DG} S\, [\{q\text{-isom.}\}^{-1}]$

Ex: $S = S_0 \rightsquigarrow$ usual derived category.

- $D^f(S) \overset{vs}{\hookrightarrow} D^{perf}(S)$ ⚠ (for our S: the same, but non-trivial!)

Def: $D^{per}(S) := \mathrm{Thick}(S) \subseteq \mathcal{D}(S)$, perfect dg-modules.

because S noeth. $\rightsquigarrow$ $D^{per}(S) \subseteq \mathcal{D}^f(S)$ ✓

Note: $D^{per}(S) =$ the compact objects of $\mathcal{D}(S)$, i.e.:

$$= \{ P \mid \mathrm{Hom}(P, \textstyle\bigoplus X_i) \cong \bigoplus_i \mathrm{Hom}(P, X_i) \}$$

• $\mathcal{D}^{per}(S)$ consists of sumands of P with:

$$0 \subseteq P_0 \subseteq P_1 \subseteq P_2 \subseteq \dots \subseteq P_m = P \ , \quad P_{i+1}/P_i \cong \bigoplus_{i=-m_1}^{m_2} S[i]^{\oplus k_i}$$

(thus: proj in Gr-mod, but <u>not</u> in dg-mod...)

↳ If $S = S_0$, this means: $P^\cdot$ perfect $\iff$ b'ded cplx of f.g. proj.

I. Homological supports

$\mathfrak{p} \in \operatorname{Spec}^* S$ a graded (= homogeneous) ideal.

$$\begin{tikzcd}
\operatorname{Mod}_{DG} S \arrow[r, "(-)_{(\mathfrak{p})}"] \arrow[d] \arrow[dr, phantom, "\text{exact functor}"] & \operatorname{Mod}_{DG} S_{(\mathfrak{p})} \arrow[d] \\
D(S) \arrow[r, "(-)_{(\mathfrak{p})}"] & \mathcal{D}(S_{(\mathfrak{p})})
\end{tikzcd}$$

[graded loc. Srin's talk: $S_{\mathfrak{p}}$

• **Def**: $\operatorname{Supp}^h(M) := \{\mathfrak{p} \mid M_{(\mathfrak{p})} \neq 0 \text{ in } \mathcal{D}(S_{(\mathfrak{p})})\}$.

• Observation:
$$\operatorname{Supp}^h M = \operatorname{Supp} H^*(M).$$

Lemma: Let $M \in D^f(S)$. If $\mathfrak{p} \in \operatorname{Spec}^* S$,
$$\mathfrak{p} \in \operatorname{Supp}^h(M) \iff k(\mathfrak{p}) \otimes^L_S M \neq 0 \text{ in } D(-)$$

Pf: use Nakayamas Lemma ... derive!

Theorem (Hopkins, Neeman, Carlson–Iyengar)
Let $M, N \in \mathcal{D}^{perf}(S)$. Then $\operatorname{Supp}^h M \subseteq \operatorname{Supp}^h N \iff M \in \operatorname{Thick}(N)$.

$\Leftarrow$ ✓ easy, using the "usual" properties of support

Difficult part: "$\Rightarrow$".
Uses the tensor product!!!

Remark:

- If $(M \otimes_S N)^{\#} = M^{\#} \otimes_S N^{\#}$, $d(m \otimes n) = d(m) \otimes n + (-1)^{|m|} m \otimes d(n)$
  $\rightsquigarrow$ $M \otimes_S N$ of dg-modules / S. $\rightsquigarrow M \overset{L}{\underset{S}{\otimes}} N$.
- If $M, N \in \mathcal{D}^{per}(S) \Longrightarrow M \otimes^L_S N \in \mathcal{D}^{per}(S)$!

In fact, N perfect $\rightsquigarrow$
$$\begin{cases} 0 = P_0 \subseteq P_1 \subseteq \dots P_m \xleftarrow{\oplus} N \\ P_0 = \bigoplus S[i]^{l_i} \\ 0 \to P_i \to P_{i+1} \to \bigoplus S[j]^{l_j} \to 0 \end{cases}$$

$\rightsquigarrow$ $M \otimes^L_S N \underset{\substack{\uparrow \\ N \text{ semifree}}}{=} M \otimes_S N$; $0 \subseteq \underline{M \otimes_S P_0} \subseteq M \otimes_S P_1 \subseteq \dots M \otimes_S P_m \xleftarrow{\oplus} M \otimes N$

$$\begin{cases} M \otimes_S P_0 \cong \bigoplus \Sigma^i M^{\oplus l_i} \\ 0 \to M \otimes P_i \to M \otimes P_{i+1} \to \bigoplus M[j]^{l_j} \to 0 \end{cases}$$

shifted copies of M!

this becomes a Δ in $\mathcal{D}(S)$

So this gives a recipe of how to construct $M \otimes N$ out of M via Δs. $\rightsquigarrow$ $M \otimes^L_S N \in \text{Thick}(M)$.

Also: $\Longrightarrow$ thick subcategories are automatically $\otimes$-ideals!

This gives an idea of how the $\otimes$ enters the proof ....

II. More gr-comm. algebra

- ($\mathbb{Z}$-) graded prime $\equiv$ graded & prime (or: graded-prime)
- graded max. ideal: max. among graded ideals
- S/m graded field, see [Bruns-Herzog, C-M-rings §1.5]

$$\circledast \quad S/m = K = K^0 \quad\text{or}\quad S/m = K^0[t^{\pm 1}],\ |t|>0 \quad \text{graded field, NOT field!}$$

- Krull dimension (alg. version of the dim. of alg. varieties)

$$\dim S \underset{g}{:=} \sup\{ n \mid \exists\ p_0 \subsetneq p_1 \subsetneq \dots \subsetneq p_n \text{ primes}\}.$$

usual ungraded comm. ring

Graded mimic of this:

$$\dim^* S := \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ graded primes}.$$

- ⚠ fact $\dim S$, $\dim^* S$ differ at most by 1. (Cfr $\circledast$!)
- When $\dim S < \infty$?
  - → If S is a f.g. algebra over a field ✓
  - → If S is graded local ✓ ($\Leftarrow$ Krull's principal ideal theorem ...)
- If S is graded comm. noetherian, then:

$$\mathcal{N} := \sqrt{(0)} = \{\text{all nilpotent elements}\}, \text{ the nilradical}$$

automatically a graded ideal

- ([Bruns-Herzog]): more generally, if $\mathfrak{p}$ is a minimal ordinary prime ideal of S, then it is automatically graded! (E.g.: $\mathcal{N} = \bigcap_{\mathfrak{p} \text{ min prime}} \mathfrak{p}$).

III. Tensor nilpotence

- $M \in \mathcal{D}^{per}(S)$. Then, $M = 0 \iff \mathrm{Supp}^h M = \emptyset$

$$\iff M \otimes^L_S \underbrace{k(p)}_{S_{(p)}/(p)\cdot S_{(p)}} = 0 \quad \forall p \text{ gr. prime}$$

"Almost" analog for maps:

<u>Tensor-Nilpotence Theorem</u> (or: a corollary)

> Let $S \xrightarrow{f} F$ be a map in $\mathcal{D}^{per}(S)$.
> If $f \otimes^L_S k(p) = 0 \ \forall p$, then $f^{\otimes^L n}: S = S^{\otimes^L n} \to F^{\otimes^L n}$,
> for some $n > 0$.

<u>Pf</u> (ideas)

Step I: reduce to the case when S is a domain

- WLOG: F is semi-free
- suppose that $f: S \to F$ is s.t. $f \otimes k(p) = 0 \ \forall p$.

$\Rightarrow \forall p$ minimal, $\bar{f}: S/p \longrightarrow F/_{F \cdot p}$

$\Rightarrow \bar{f} \otimes^L_{S/p} k(q) = 0 \quad \forall q \in \mathrm{Spec}\, S/p$

True for domains $\Rightarrow \bar{f}^{\otimes n} = 0$ for some n.

Now, f is determined by $x := f(1) \in F^0$, and $d(1)$ in S implies also: $d(x) = 0$ in F^0: a zero cocycle!

Now: $f^{\otimes n}: S \longrightarrow F^{\otimes n} =: G$

$\qquad 1 \longmapsto x \otimes x \cdots \otimes x =: y$

Since $\bar{f}^{\otimes n} = 0$, then $y = \underset{\in G \cdot p}{y_1} + \underset{\text{a coboundary}}{y_2}$ ← killed by p up to coboundary!

- Do that $\forall\, \mathfrak{p}$ minimal prime, $\exists$ fin. many such

$\rightsquigarrow \exists\, n' : f^{\otimes n'}(1) \in F^{\otimes n'} \cdot N$ + coboundary

$\rightsquigarrow \exists\, n'' : f^{\otimes n''}(1) =$ coboundary

$\rightsquigarrow f^{\otimes n''} = 0$ as a map in $\mathcal{D}(S)$.

<u>Step II</u>: Induction on $\dim^* S$, if finite!

Step III: What if $\dim^* S = \infty$?

$\rightsquigarrow$ localize at some $\mathfrak{p}$ $\rightsquigarrow$ back to step II.

IV. Proof of Theorem.

Let $M, N \in \mathcal{D}^{perf}(S)$, $\operatorname{Supp}^h M \subseteq \operatorname{Supp}^h N$.

We have: $S \longrightarrow \mathbb{R}\mathcal{H}om(N,N)$ (think: ordinary ring S, ordinary module N)

$1 \longmapsto id_N$

$$\overset{M \otimes^L_S -}{\rightsquigarrow} \quad F \xrightarrow{f} M \xrightarrow{g} M \otimes^L_S \mathbb{R}\mathcal{H}om(N,N) \quad \Big\} \text{ in } \mathcal{D}^{perf}(S)$$

complete to a Δ!

$\mathbb{R}\mathcal{H}om$ also stays inside!

$\rightsquigarrow$ prove that $f \otimes k(\mathfrak{p}) = 0 \quad \forall\, \mathfrak{p} \in \operatorname{Supp}^h(M)$.

(<u>Rmk</u>: f iso in $\mathcal{D}(-) \iff f$ iso in H^*

But: f zero — $\underset{\not\Leftarrow}{\Longrightarrow}$ f zero —

by far!

<u>BUT</u>: over a field, the problem vanishes!)

$\overset{\otimes\text{-nilp.}}{\Longrightarrow}$ $M \otimes f^{\otimes n} = 0$ for some n, & $\operatorname{Cone} f^{\otimes n} \in \operatorname{Thick}(N)$

only have to consider $\mathfrak{p} \in \operatorname{Supp}^h(M)$, are already $= 0$ otherwise!

$M^{\otimes(n+1)} \leq_{\oplus} (\operatorname{Cone} f^{\otimes n}) \otimes M \in \operatorname{Thick}(N) \Longrightarrow M \in \operatorname{Thick} N$. $\rightsquigarrow \square$